Elasticity: Term Paper

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I. Abstract

This research was conducted in order to experimentally test certain components of the theory of elasticity. The theory was tested through a variety of extensions, bends, compressions, and shears. The resulting strain was measured using rulers, micrometers, calipers, and strain gauges.

II. Theory

For a simple extension, the Young's modulus (E) of a material can be found using equation 5.2 in *Theory of Elasticity*,

$$u_{zz} = p / E \quad (1)$$

where u_{zz} is the longitudinal strain or relative change in length, given by the longitudinal displacement (Δz) per unit unstrained height (h_0), and p is the longitudinal stress, given by normal force (F_N) per unit cross-sectional area (A) [1].

$$u_{zz} = \Delta z / h_0 \quad (2)$$

$$p = F_N / A \quad (3)$$

The Poisson ratio of the material is then given by equation 5.4 in *Theory of Elasticity*,

$$u_{xx} = -\sigma \times u_{zz} \quad (4)$$

where u_{xx} is transverse strain, given by the relative change in width

$$u_{xx} = \Delta x / w_0 \quad (5)$$

and u_{zz} is longitudinal strain, given by Equation 2 [1]. These experimental variables are shown in Figures 1 and 2 for a simple extension of a wire and foam block respectively.

For a pure shear, the shear modulus (μ) of a material can be found using the following equation,

$$p = \mu \times \varepsilon \qquad (6)$$

where p is the stress given by tangential force (F_T) per unit cross-sectional area (A), and ε is the shear strain given by the displacement of the surface to which the force is applied (Δx) per unit height (h).

$$p = F_T / A \qquad (7)$$

$$\varepsilon = \Delta x / h \qquad (8)$$

Figure 3 depicts an example of a shear strain showing these experimental variables.

For a bending rod, the Young's modulus (E) of a material is given by,

$$u_{zz} = [-(3z_0) / (Ea^2b)]\, F \qquad (9)$$

where u_{zz} is the relative change in length of a point on the surface of the rod, z_0 is the initial length, a is the thickness of the rod, b is the width, and F is the applied force. These experimental variables for a bending rod are demonstrated in Figure 4.

For a standing rod, the lateral strain (u_{xx}) as the rod expands due to the force of its own weight is given by

$$u_{xx} = (\sigma\rho g / E)\,(l - z) \qquad (10)$$

where σ is the Poisson ratio, ρ is the density, and E is the Young's modulus of the material. The length of the rod is given by *l*, and the distance from the point where the strain is being measured to the base of the rod is z. Finally, g is gravitational acceleration. A depiction of a standing rod labeled with the experimental variables *l* and z is given in Figure 5.

For elastic waves, the velocities of the transverse (c_t) and longitudinal (c_l) waves in a medium are given by

$$c_t = \sqrt{[E (1 - \sigma) / \rho (1 + \sigma) (1 - 2\sigma)]} \text{ and } c_l = \sqrt{[E / 2\rho (1 + \sigma)]} \quad (11)$$

where E is the Young's modulus of the material, σ is the Poisson ratio, and ρ is the density [1].

III. Experimental Details

a. Simple Extension of Wire

A thin copper, a thick copper, and a steel wire were each hung from height of 60 ft. The wires extended from the hanging point to the ground, unstrained. For each wire, the diameter was measured using a micrometer to find the cross-sectional area. A small piece of tape was used to mark an arbitrary location on each wire near the bottom (Figure 1). The height of the tape from the ground was recorded and subtracted from the full length of the wire (60 ft) in order to obtain h_0. A ruler was mounted behind each wire so that the location of the tape relative to the ruler could be measured. This initial measurement was recorded as zero displacement. A loop was made on each wire, below the tape, from which weights would be hung. Force was applied, using weights or a spring gauge, in the longitudinal direction. The amount of force applied was increased incrementally and was recorded in Newtons. For each amount of force, the new location of the tape along the ruler was recorded. The zero mark was subtracted from this value to give the displacement of the wire Δz. The stress and strain were calculated using the data recorded and equations 2 and 3 respectively. Table 1 gives the types of wires used and their cross-sectional areas.

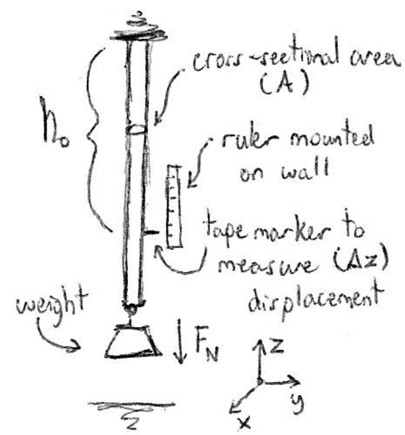

Figure 1. Diagram of the experimental setup for the simple extension of wire.

Table 1. Types of wires used in simple extension experiment and their cross-sectional areas.

	Cross-sectional area (cm^2)	**E (GPa)**
Thin copper	8.96×10^{-4}	76
Thick copper	4.23×10^{-3}	78
Steel	7.42×10^{-4}	1.8×10^{2}

b. Simple Extension of Foam Blocks

Nine foam blocks of varying dimensions were glued to a board. The board was clamped so that the blocks hung upside down. For each block, the initial height (h_0), width (w_0), length (L), and cross-sectional area (A) were measured using rulers and calipers (Table 2). A thin plexiglass plate with an equivalent cross-sectional area was attached to the top face of each block. A ruler was secured so that the longitudinal displacement of each foam block could be recorded. A spring gauge was hung from a loop of string that was attached to the center of each plexiglass square. The spring gauge was used to apply different amounts of longitudinal force (F_N) to each block. For each amount of force applied, the transverse displacement (Δx), measured using calipers, and the vertical displacement (Δz), along the ruler, were recorded in millimeters. The longitudinal strain (u_{zz}), stress (p), and transverse strain (u_{xx}) were calculated from this experimental data using equations 2, 3, and 5 respectively. This experimental arrangement is displayed in Figure 2.

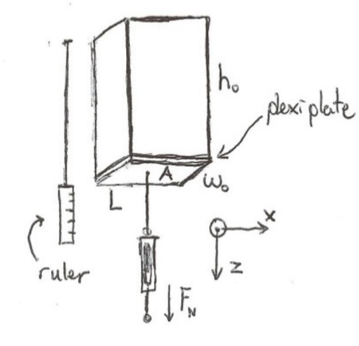

Figure 2. Diagram of the experimental setup for the simple extension of foam blocks.

Table 2. The dimensions in millimeters, the Young's modulus (E) in kilopascals, and the Poisson ratio (σ) are given for each foam block.

Block	L (mm)	w_0 (mm)	h_0 (mm)	E (kPa)	σ
1	19.6	20.8	19.9	195	0.876
2	21.7	19.0	43.7	177	0.759
3	19.4	18.5	83.2	181	0.733
4	31.1	31.4	23.5	241	0.950
5	31.9	31.1	42.6	443	1.908
6	32.2	32.4	81.5	260	1.200
7	38.4	41.5	21.0	144	0.538
8	40.5	41.7	42.7	372	1.020
9	41.1	40.5	79.4	150	0.782

c. Pure Shear of Foam Blocks

Eleven foam blocks of various dimensions were glued to a board, each with a plexiglass square of equivalent cross-sectional area glued to its top face (Figure 3). The dimensions of each block, measured using rulers and calipers, are given in Table 3, including height (h). The cross-sectional area of each block (A) was calculated from the measurements of length and width. Rulers were mounted on the board so that the horizontal position of each block relative to its ruler could be determined. A spring gauge was hooked through a loop of string attached tangential to the top face of each block in order to apply a tangential force (F_T). As the amount of applied force was incremented, the resulting horizontal displacement of the top face (Δx) of each block along the ruler was recorded. The stress (p) and shear strain (ε) were calculated using the experimental measurements and equations 7 and 8 respectively.

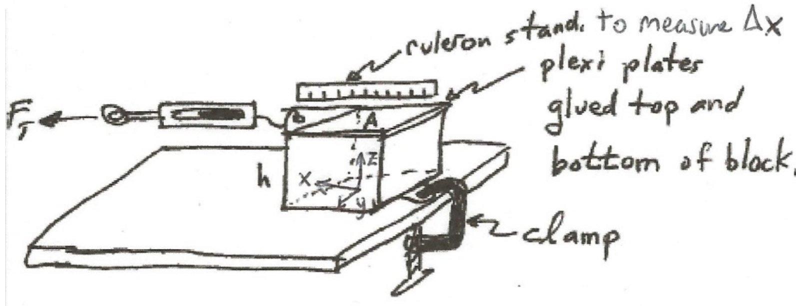

Figure 3. Diagram of the experimental setup for the pure shear of foam blocks, where F_T is the tangential force, h is the height of the block, A is the cross-sectional area, Δx is the change in position of the top face of the block along the ruler, and the coordinate axes are given.

Table 3. The length, width, and height (h) of the foam blocks in millimeters and the shear modulus (μ) in kilopascals are given for the eleven blocks in the pure shear experiment.

Block	Length (mm)	Width (mm)	h (mm)	μ (kPa)
1	39.5	39.8	39.9	22.0
2	40.5	40.5	20.4	43.1
3	40.6	40.2	10.5	121
4	29.9	30.8	40.9	32.6
5	29.8	30.8	21.1	55.6
6	29.4	31.3	9.2	86.6
7	22.2	22.4	39.5	29.3
8	22.3	20.1	23.0	31.1
9	23.0	21.6	9.8	38.7
10	12.2	43.0	20.8	38.3
11	41.4	11.9	22.8	16.3

d. Strain in a Bending Rod

A strain gauge was mounted on an aluminum bar with the grid parallel to the length of the rod (Figure 4). One end of the bar was clamped to a counter. The distance from the clamp to the other end of the rod was recorded as initial unstrained length (z_0). A spring gauge was attached to the unclamped end in order to apply force down or up on the bar, causing it to bend. The leads connected the strain gauge to a Wheatstone bridge, and the bridge was connected to a multimeter. As different amounts of force were applied, the voltage was recorded. Downward force resulted in a negative change in voltage and upward, a positive change. The strain for each amount of applied force was calculated using

$$\varepsilon=4\Delta V/(10V*G) \quad (12)$$

where G was the gauge factor, 2.03, and ΔV was the change in voltage read on the multimeter. The rod was then clamped to a new position and the process repeated for two other values of z_0. This experiment was repeated for two rods of different width (b) and thickness (a) given in Table 4.

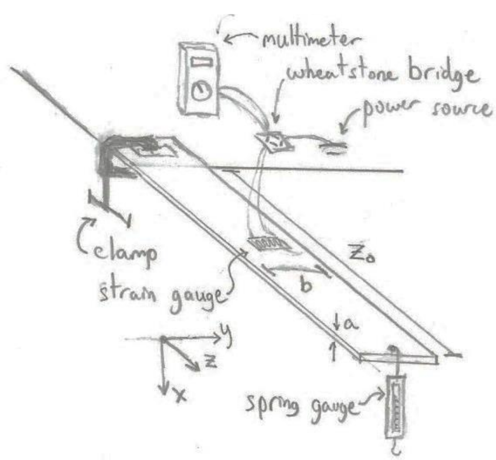

Figure 4. Diagram of the experimental setup for bending a metal rod.

Table 4. The unstrained length (z_0) in centimeters, thickness (a) in millimeters, width (b) in centimeters, and Young's modulus (E) in gigapascals for each rod.

Rod	z_0 (cm)	a (mm)	b (cm)	E (GPa)

1	29.0	2.4	2.5	1.5×10^{2}
1	25.0	2.4	2.5	1.3×10^{2}
1	19.5	2.4	2.5	89
2	38.1	1.7	10.1	1.2×10^{2}
2	35.1	1.7	10.1	1.1×10^{2}
2	29.0	1.7	10.1	91

e. Strain in a Standing Rod

A strain gauge was mounted on a steel rod, near the bottom, so that the grid was perpendicular to the length of the rod. The strain gauge was connected to an amplifier with a gain of 1000, which was strapped to the rod. The amplifier was then connected to a power source and to a chart recorder. As the rod was raised off the ground and placed down again, the change in voltage on the chart recorder indicated a change in strain in the base of the rod as it contracted and expanded. For runs one and two, the chart recorder wrote as the rod was quickly and carefully lifted and set down several times. Then the process was repeated with weight added to the rod so that the effect was amplified. The length (*l*) of the rod was measured to be 91.4 cm, and the distance from the strain gauge to the base of the rod was 26.7 cm. This experimental arrangement is shown in Figure 5, and a circuit diagram of the amplifier is given in Figure 6.

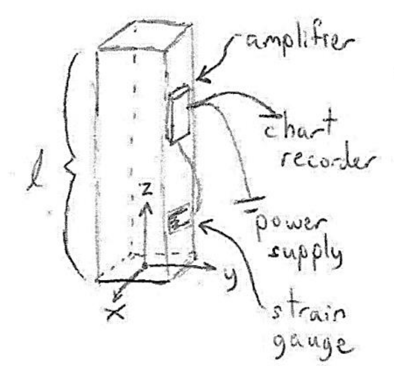

Figure 5. This is an image of the standing rod length *l*, showing the strain gauge positioned a distance of z above the base of the rod. The strain gauge is connected to an amplifier strapped to the rod, which connects to the power supply and chart recorder.

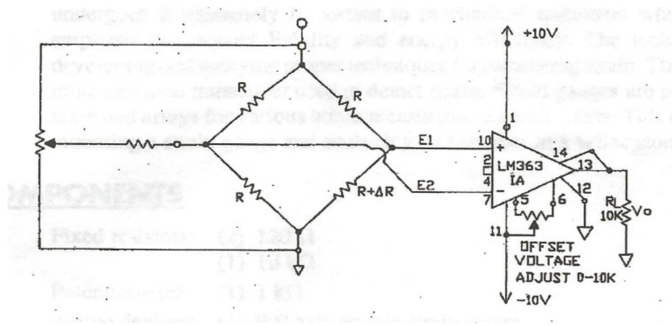

Figure 6. A circuit diagram of the amplifier that was connected to the strain gauge.

This experiment was repeated for a plexiglass rod, with length (*l*) 184.2 cm and position of strain gauge (z) 2.5 cm. Three runs were recorded without added weight, and one with added weight. For both the metal and plexiglass rods, the voltage reading of the chart recorder drifted, so the rods had to be quickly lifted and lowered to decrease the effect of the drift on the recorded change in voltage. Figure 7 gives an example of the chart recorder read out.

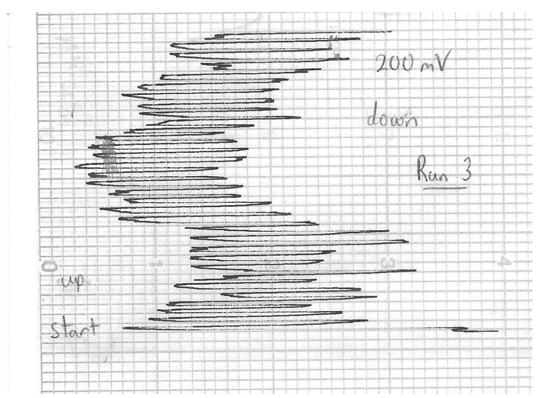

Figure 7. The chart recorder output for run three of the experiment with the plexiglass rod.

From the chart recorder output for each run, the change in voltage for each time the rod was lifted and set down was measured, and the mean and standard deviation of this data were found. The strain (u_{xx}) for each run was calculated from the mean change in voltage (ΔV) using Equation 12 (Tables 5 and 6).

f. Speculative Section – Sound waves

A transducer could be used to convert an electrical signal from a pulse generator to a mechanical signal in a sample, such as plexiglass. A transducer on the other end of the material could send an electrical signal to an oscilloscope, and the time it takes for the signal to travel through the material could be measured in order to calculate the speed of the longitudinal waves. Then, this process would be repeated but with the mechanical signal at a different angle of incidence. Again the time of travel would be measured. This wave can be split into horizontal and vertical components, one along the direction of the incident signal and one perpendicular to this direction. Knowing the longitudinal wave speed from the first part of the experiment and resultant wave speed just measured, the transverse speed could be calculated.

II. Results

a. Simple Extension of Wire

The graph below (Figure 8) shows a plot of stress vs. strain for the wire experiment. The Young's moduli (E), determined from the linear fits and equation 1, for each wire are given in Table 1. The slopes of the two copper wires were very similar, but the steel wire has a much smaller slope. There is a wave in the data plotted for the steel wire.

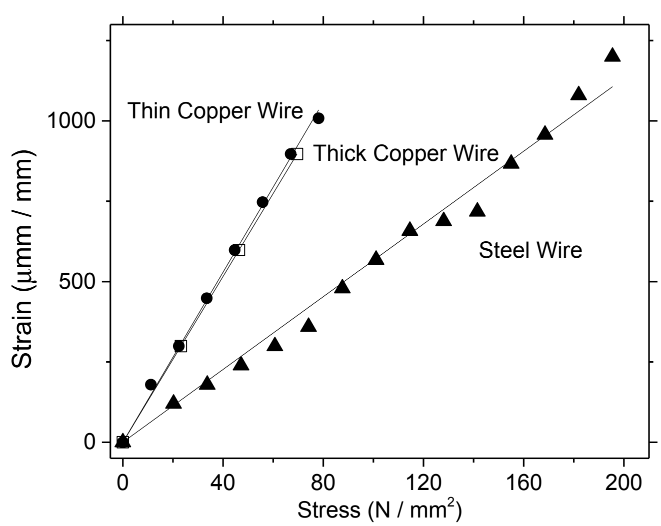

Figure 8. Plot of stress (p) vs. strain (u_{zz}) for the long wires.

b. Simple Extension of Foam Blocks

The two graphs below (Figures 9 and 10) show stress vs. longitudinal strain for the simple extension experiment with foam blocks. The first graph (Figure 9) gives the results for blocks one through three, and the second graph (Figure 10) displays the data for blocks four through nine. The two blocks indicated by open circles, blocks three and six, seemed to be a different type of foam than the other seven blocks. The Young's moduli (E) for each block, given by the linear fit and Equation 1 are given in Table 2. The Young's moduli of blocks four, five, six, and eight were much larger than those of the other blocks.

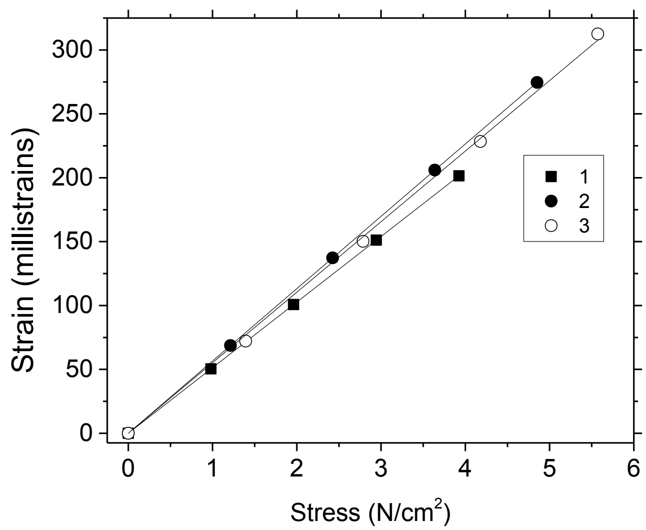

Figure 9. Plot of stress (p) vs. strain (u_{zz}) for foam blocks one through three.

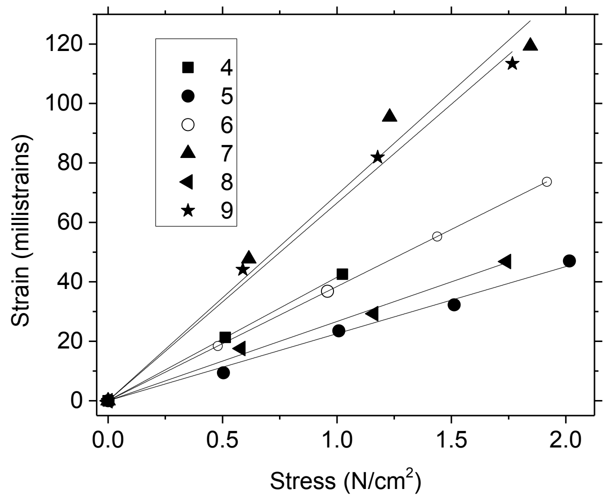

Figure 10. Plot of stress (p) vs. longitudinal strain (u_{zz}) for foam blocks four through six.

The plot of longitudinal strain vs. the negative of the transverse strain for the same foam block experiment is shown below (Figures 11 and 12). Blocks three and six, the second type of foam, are designated by open circles. The Poisson ratios for each block (σ), given in the Table 2, were found from the linear fit of each line using Equation 4. The Poisson ratio values are all greater than 0.5.

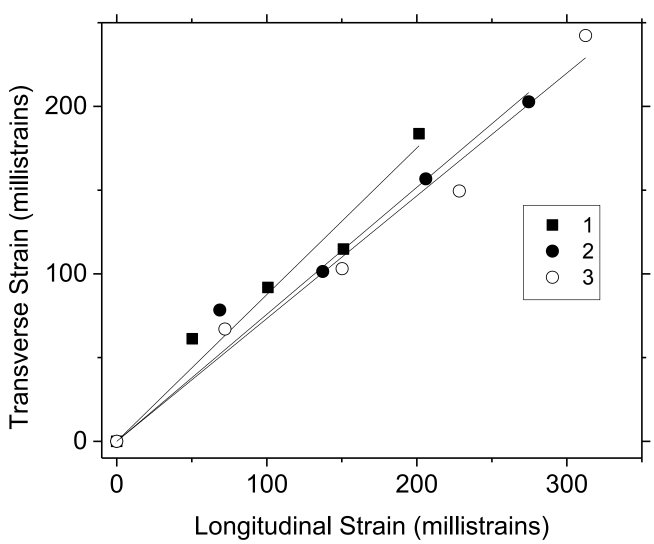

Figure 11. Plot of longitudinal strain (u_{zz}) vs. transverse strain (u_{xx}) for blocks one through three.

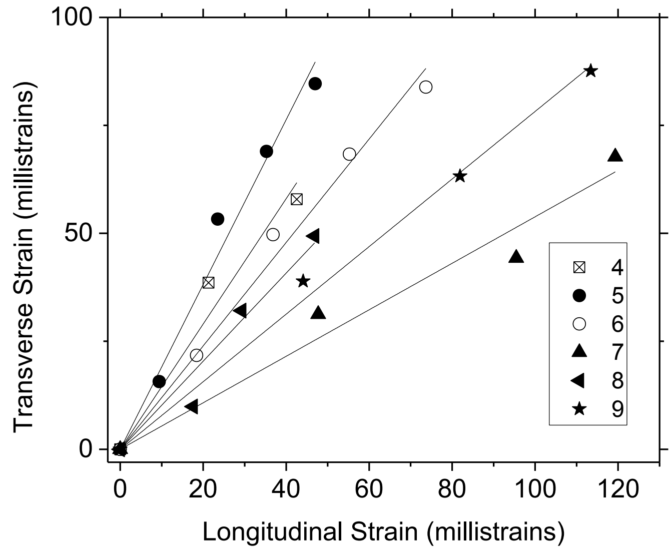

Figure 12. Plot of longitudinal strain (u_{zz}) vs. transverse strain (u_{xx}) for blocks four through nine.

c. Pure Shear of Foam Blocks

The graphs below are plots of stress vs. strain for the shear strain experiment using foam blocks. The data for blocks one through six are plotted in the first graph (Figure 13), and the data for blocks seven through eleven are plotted in the second graph (Figure 14). The blocks indicated by the open shapes (blocks 3, 6, 10, and 11) did not have pure shear; some vertical displacement still occurred (Figure 15). The shear moduli for each block, given in Table 3, were found from the linear fits using Equation 6.

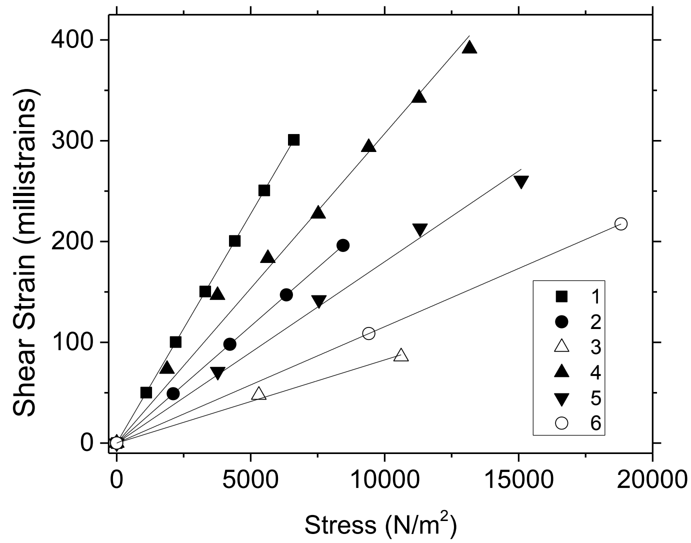

Figure 13. Plot of stress (p) vs. shear strain (ε) for foam blocks one through six.

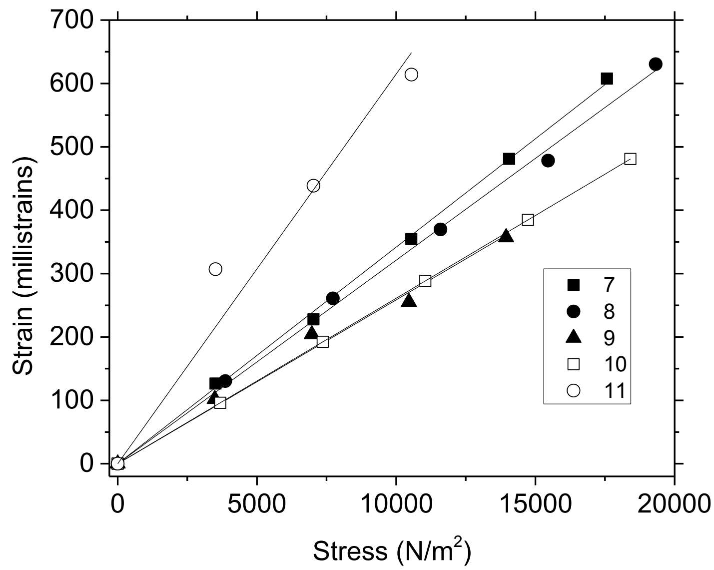

Figure 14. Plot of stress (p) vs. shear strain (ε) for foam blocks seven through eleven.

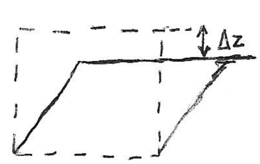

Figure 15. The dashed line shows the initial, unstrained position of the foam block. The solid line shows the position of the foam block, and the resulting vertical displacement Δz, when a tangential force was applied. This behavior was demonstrated by blocks 3, 6, 10, and 11.

d. Strain in a Bending Rod

The graphs below (Figures 16 and 17) are plots of force vs. strain for rods one and two respectively. The positive force and strain correspond to an upward force on the rod, while the negative force and strain correspond to a downward force on the rod. The relationship is linear. The Young's moduli for each trial, given in Table 4, were calculated from the dimensions of each rod and the corresponding linear fits using Equation 9.

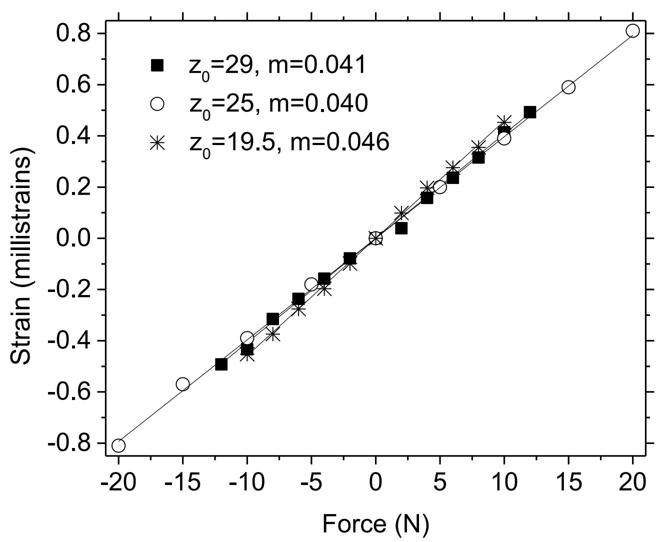

Figure 16. Plot of force in Newtons vs. strain in millistrains for rod one in the bending rod experiment, for three initial lengths (z_0) in centimeters. Negative strain corresponds to a negative (downward) force on the rod, and positive strain, to a positive (upward) force on the rod. The slopes (m) of the linear fits for each value of z_0 are given.

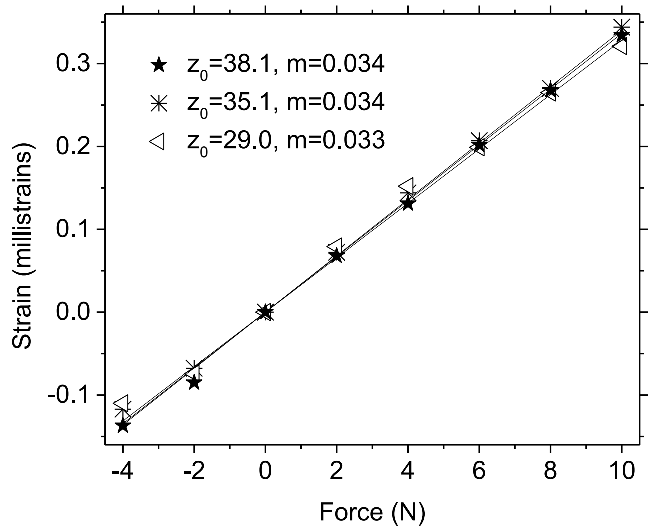

Figure 17. Plot of force in Newtons vs. strain in millistrains for rod two of the bending rod experiment, with three initial lengths (z_0) and the slope of the linear fit (m) for each.

e. Strain in a Standing Rod

For the upright steel rod, the average strain calculated for each run and the corresponding standard deviation for that run are listed in Table 5 below. Run 3 was conducted with additional weight on the rod. The strain recorded in run 3 was more than double the strain in runs 1 and 2.

For the plexiglass rod, the average strain and standard deviation for the four runs are given in Table 6. The fourth run was conducted with additional weight on the rod. Strain was much greater in the plexiglass rod than in the steel rod.

Table 5. The strain (u_{xx}) in microstrains calculated from the average change in potential and the standard deviation (σ) of the measured changes in potential for runs one through three with the steel rod. The third run was carried out with weight added to the rod.

Run	u_{xx} (microstrains)	Standard deviation
1	0.17	0.64
2	0.14	0.20
3	0.47	0.65

Table 6. The strain (u_{xx}) in millistrains calculated from the average change in potential and the standard deviation of the measured changes in potential for runs one through four with the plexiglass rod. Run four was conducted with additional weight on the rod.

Run	u_{xx} (microstrains)	Standard deviation
1	6.83	4.48
2	6.53	4.82
3	5.43	5.46
4	19.0	10.05

IV. Conclusion

a. Simple Extension of a Wire

The Young's modulus for copper is 117 GPa, which is about 1.5 times the value determined by the experiment, given in Table 1, for the thin and thick copper. This error may have been a result of inaccurate measurement of the displacement of the wire along the ruler. The ruler should be fixed at a position just behind the wire, so that the position of the tape can be more accurately read. The wire may also have slipped from its initial position as force was applied. The wire must be securely tied from where it is hung. The additional slipping would make the wire appear to be more elastic, and therefore have a lower Young's modulus.

The Young's modulus for steel is 200 GPa. The Young's modulus (Table 1) calculated from the experimental data was close to the true value, with a 10% error. The curve in the data plotted for the steel rod (Figure 8) may have been a result of rounding. The displacement for each applied force was a fraction of a millimeter, but was measured on a ruler, where the smallest demarcation was millimeters. The measurement may have alternately been rounded down and up, resulting in the curve in the graph.

b. Simple Extension of Foam Blocks

For low density polyurethane foam, the Young's modulus is approximately 0.1 MPa. The Young's moduli of foam blocks 1, 2, 3, 7, and 9 (Table 2) calculated from the experimental data and theory came close to this value. However, blocks 4, 5, 6, and 8 had Young's moduli two to four times greater than the expected value. The dimensions of these blocks were similar to those of blocks 1, 2, 3, 7, and 9. The error may have arisen because the glue affected the Young's modulus of certain portions of the foam. For a few of the blocks, the glue between the foam and the board or between the foam and the plexiglass had seeped into the foam, making the top and bottom faces of the blocks very stiff. This would affect the results of the experiment, making those blocks appear to have a larger Young's modulus.

Blocks 3 and 6 were thought to be a different kind of foam. However, because of the similar results obtained, they appear to actually be the same kind of foam as the other blocks.

The Poisson ratio for polyurethane foam is approximately 0.3. However, the experimental Poisson ratio values were all greater than 0.5. This is an unphysical situation, since the Poisson ratio can only take on values between -1 and ½. This strange outcome could have been a result of the force not being uniformly applied to the bottom face of the block. The plexiglass squares did not have exactly the same cross-sectional area as the foam blocks, so the edges of the blocks were not connected to the squares. Thus the center of the foam block experienced a normal force, while the outside edges of the block experienced a weaker tangential force. The theory did not model this situation properly. This problem was exacerbated by the fact that the glue could not hold up under the stress, so the plate continued to detach from the foam as more force was applied. As a result, the observed transverse displacement was much greater than it would have been if a uniform normal force were applied to the bottom face. Also the expectation of a large transverse displacement may have led to the inaccurate measurement of the true displacement.

c. Pure Shear of Foam Blocks

Most blocks had shear moduli between 22 and 56 kPa, but extreme shear moduli values were given by blocks 3, 6, and 11 (Figure 13 and 14). Since those blocks, as well as block 10, did not demonstrate true shear strain, the theory for pure shear did not fully apply. Vertical displacement occurred for those blocks (Figure 15). Those particular blocks should be removed from the experiment. Blocks 3 and 6 were very short compared to the other blocks, and blocks 10 and 11 were more rectangular shaped than the other blocks. The blocks that were more cube-shaped had better results. The theory may be better tested by using a different material, such as rubber.

d. Strain in a Bending Rod

For aluminum, Young's modulus is 70 GPa. All the experimental values for the Young's modulus were about twice this value (Table 4). The modulus should have been constant; however, the value varied between each trial. The mean Young's modulus was 114 with a standard deviation of 23.4. For each rod, the experimental values of the Young's moduli, found from Equation 8, decreased with a decrease in the initial length of the rod. This implies that the slopes of the linear fits for each rod (Figures 16 and 17) changed by a smaller factor than the initial length, as the other dimensions were held constant, though they should have changed by the same factor. Thus as length decreased, a larger than expected change in voltage was recorded for each applied force. This may be due to the optimism of the experimenters, looking for a larger change in voltage than was present, resulting in inaccurate measurement. It may also be that the curvature of the rods could not be approximated by the arc of a circle as in the theory, since they did not bend uniformly.

The slopes of the lines for each rod theoretically would decrease as the initial length decreased and the other dimensions were held constant. However as seen in Figure 16, the third slope corresponding to the smallest initial length is the largest, though it should be the smallest. In Figure 17, the slopes of the lines decreased with decreasing values of initial length, which agrees with the theory. The slopes of the lines for rod 1 are greater than those for rod 2, as expected from the theory, since the constant of proportionality in Equation 9 for a given initial length is larger for rod 1 than rod 2.

e. Strain in a Standing Rod

Based on Equation 10 and the constants provided in Table 7, the strain in the steel rod would theoretically have been 0.075 microstrains. The experimental values of 0.168 and 0.138 microstrains have percent errors of 124% and 84% respectively. The error is likely due mostly to the drift and the noise in the chart recorder output. The noise was approximately 1 mV, and the measured changes in voltage were on the order of 0.5 mV. Another possible error is due to the amplifier, which may have had a greater gain than 1000.

By Equation 10 and the constants in Table 7 below, the strain in the plexiglass rod would theoretically have been 2.4 microstrains. Experimentally, the values of 6.83, 6.53, and 5.43 have percent errors of 185%, 172%, and 126%. Again, the drift in the chart recorder output and the gain due to the amplifier are sources of error. Also, rounding up due to an expectation for a larger change in voltage may have caused the measurements to be greater than the true value.

Table 7. The Young's modulus (E), Poisson ratio (σ), density (ρ), length of the rod (*l*), position of the strain gauge (z), and calculated strain (u_{xx}) for the steel and plexiglass rods.

Material	**E (GPa)**	**σ**	**ρ (kg/m^3)**	***l* (cm)**	**z (cm)**	**u_{xx} (microstrains)**
Steel	200	0.30	7850	91.4	26.7	0.075
Plexiglass	3.10	0.35	1180	184.2	2.5	2.4

V. References

1. Landau, L. D., & Lifshitz, E. M. (1986). *Theory of Elasticity* (3rd ed.). Oxford: Reed Educational and Professional Publishing Ltd.